

Lecture 14 (spin) Deutsch algorithm and its optical implementation

- Deutsch algorithm
- Deutsch-Jozsa algorithm
- optical implementation of Deutsch's algorithm

quantum bit (qubit)

Quantum gate

quantum circuit

quantum function

quantum algorithm

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]}$$

$f(a) \in \mathcal{O}(a)$ $\mathcal{O}(a) \xrightarrow{f} f(a)$
 $\mathcal{O}(f(a)) = \mathcal{O}(\mathcal{O}(a))$
 $= \{a\}$ $f(a) \xrightarrow{\mathcal{O}} a$

$$\hat{H}_{int}(\hat{\sigma}_+, \hat{\sigma}_-, \hat{a}^\dagger, \hat{a}) \rightarrow \text{Hermitian}$$

① observable (eigenvalue) of $\hat{H}$ is real.

② propagator $\hat{U} = e^{-i\hat{H}\Delta t/\hbar}$
is unitary (one-to-one mapping)

$$\hat{U}^\dagger = e^{i\hat{H}_{int}t/\hbar} = e^{-i\hat{H}_{int}t/\hbar}$$

$$\hat{U}^\dagger \hat{U} = e^{i\hat{H}_{int}t/\hbar - i\hat{H}_{int}t/\hbar} = 1$$

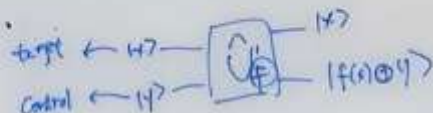
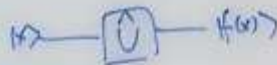
$$(\hat{U}^\dagger = \hat{U}^{-1})$$

②

quantum gates are ^{always} reversible ($I^\dagger$ exists)

classical gates are not always reversible

Constant function, $f(x) = \text{Constant}$
(Many-to-one)



NOT

In	out
1	0
0	1

 (reversible)

AND

In	out	out
1	0	0
0	1	0
0	0	0

 (irreversible)

→ reversible version

Total gate

1	0	1	1	0	1	0	1	0	1
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- Deutsch algorithm
- Deutsch-Jozsa algorithm
- optical implementation of Deutsch's algorithm

quantum algorithm $\xrightarrow{\text{recipe (construction)}}$ problem

(CONSTRAINTS)

memory inquiry

$0-100 \rightarrow 6$ classical bit
 6 qubit

$\{0,1\}^1, \{0,1\}^2$
 $\{00, 01, 10, 11\}$
 $\downarrow$
 2
 $\downarrow$
 4

$0 \rightarrow 0 \quad 0 \cdot 2^0 = 0$
 $2 \rightarrow 10 \quad 1 \cdot 2^1 + 0 \cdot 2^0 = 2$
 $5 \rightarrow 101 \quad 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5$
 $10 \rightarrow 1010 \quad 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 10$
 $64 \rightarrow 100000 \quad 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 64$
 $100 \rightarrow 110100 \quad 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 100$

$\downarrow$
 $2^6 = 128$

⑤

* local property of the classical function $f(x)$

$\rightarrow f(89), f(55)$
 $(in) = |x\rangle|y\rangle$

$\{ \text{out} \} = |x\rangle|f(x,y)\rangle$

* global property of the classical function $f(x)$

- constant mapping
- balanced mapping

$$\{0, 1\}^b \rightarrow \{0, 1\}$$

128 runs $\rightarrow$ classical computer
1 run $\rightarrow$ quantum computer (Deutsch-Jozsa algorithm)





Deutsch construction

⑤

unchanged $|10\rangle$: $f(x)$ constant mapping
 changed $|11\rangle$: $f(x)$ balanced mapping

Deutsch algorithm
 2-bit input $\{0,1\} \rightarrow \{0,1\}$
 Deutsch-Jozsa algorithm
 n -bit input $\{0,1\}^n \rightarrow \{0,1\}$
 optical implementation
 of Deutsch's algorithm

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$|a\rangle = |0\rangle \rightarrow |\psi\rangle = |0\rangle \otimes |1\rangle$
 $H_2 H_1 |\psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[(|0\rangle \otimes |1\rangle) (|0\rangle - |1\rangle) \right]$
 $= \left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right]$

⑥

$f(x)$ constant mapping
 $f(x)$ balanced mapping

$H_2 H_1 |\psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle (|f(0) \oplus 0\rangle - |f(0) \oplus 1\rangle) + |1\rangle (|f(1) \oplus 0\rangle - |f(1) \oplus 1\rangle) \right]$

Case 1 $f(0) \oplus f(1) = 0$
 $|\psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle (|0 \oplus 0\rangle - |0 \oplus 1\rangle) + |1\rangle (|0 \oplus 0\rangle - |0 \oplus 1\rangle) \right]$
 $= \left(\frac{1}{\sqrt{2}}\right)^2 \left[(|0\rangle + |1\rangle) |0\rangle - (|0\rangle + |1\rangle) |1\rangle \right]$
 $H_2 |\psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right) (|0\rangle |0\rangle - |0\rangle |1\rangle) = \left(\frac{1}{\sqrt{2}}\right) |0\rangle (|0\rangle - |1\rangle)$

Case 2 $f(0) \oplus f(1) = 1$
Case 3 $f(0) = 1, f(1) = 0$
Case 4 $f(0) = 0, f(1) = 1$

Note

$|a \oplus 0\rangle = |a\rangle$
 $|a \oplus 1\rangle = |\bar{a}\rangle$
 $|a \oplus a\rangle = |0\rangle$
 $|a \oplus \bar{a}\rangle = |1\rangle$

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
 $H^2 = I$
 $(|0\rangle \otimes |0\rangle) \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle$
 $(|0\rangle \otimes |1\rangle) \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes |1\rangle$





- Deutsch algorithm
single register $\{0,1\} \rightarrow \{0,1\}$
- Deutsch-Jozsa algorithm
multiple register $\{0,1\}^n \rightarrow \{0,1\}$
- optical implementation
of Deutsch's algorithm

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

case B $f(1) = \overline{f(0)}$ balanced

$$|\psi_3\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle_1 (|f(0) \oplus 0\rangle_2 - |f(0) \oplus 1\rangle_2) \right. \\ \left. + |1\rangle_2 (|f(1) \oplus 0\rangle_2 - |f(1) \oplus 1\rangle_2) \right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left[(|0\rangle_1 - |1\rangle_1) |f(0)\rangle_2 \right. \\ \left. - (|0\rangle_1 - |1\rangle_1) |f(1)\rangle_2 \right] \quad (|0\rangle|0\rangle \rightarrow |0\rangle|1\rangle)$$

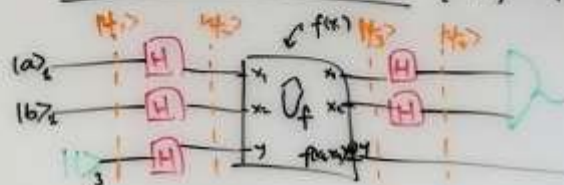
$$\xrightarrow{H_2} |\psi_4\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right)^2 \left[|1\rangle_1 |f(0)\rangle_2 - |1\rangle_1 |f(1)\rangle_2 \right] \\ = \frac{1}{\sqrt{2}} |1\rangle_1 (|f(0)\rangle_2 - |f(1)\rangle_2)$$

- ✓ - Deutsch algorithm
single register $\{0,1\} \rightarrow \{0,1\}$
- ✓ - Deutsch-Jozsa algorithm
multiple register $\{0,1\}^n \rightarrow \{0,1\}$
- optical implementation
of Deutsch's algorithm

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Deutsch-Jozsa Construction

$\{0,1\}^2 \rightarrow \{0,1\}$



① constant mapping

$f(x) = \text{constant}$

② balanced mapping

$f(00) = f(01) = 0$

$$|\psi_1\rangle = |0\rangle_1 |0\rangle_2 |1\rangle_3 \xrightarrow{H_1 H_2 H_3} |\psi_2\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 (|0\rangle + |1\rangle) (|0\rangle + |1\rangle) (|0\rangle - |1\rangle) f(11) = f(01) = 1 \\ = \left(\frac{1}{\sqrt{2}}\right)^3 \left[|00\rangle (|0\rangle - |1\rangle) \right. \\ \left. + |01\rangle (|0\rangle - |1\rangle) \right. \\ \left. + |10\rangle (|0\rangle - |1\rangle) \right. \\ \left. + |11\rangle (|0\rangle - |1\rangle) \right]$$





⑨ $H_3 = [100\rangle (|f(00)\rangle - |f(00)\rangle) + |01\rangle (|f(01)\rangle - |f(01)\rangle) + |10\rangle (|f(10)\rangle - |f(10)\rangle) + |11\rangle (|f(11)\rangle - |f(11)\rangle)]$

Case 1 Constant mapping

Note

$|a \oplus 0\rangle = |a\rangle$
 $|a \oplus 1\rangle = |\bar{a}\rangle$
 $|a \oplus a\rangle = |0\rangle$
 $|a \oplus \bar{a}\rangle = |1\rangle$

$H = H_2 \otimes H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Summary: $H^2 = I$

✓ - Deutsch's algorithm
 4-line register $\{0,1\} \rightarrow \{0,1\}$

✓ - Deutsch-Jozsa algorithm
 4-line register $\{0,1\} \rightarrow \{0,1\}$

- optimal implementation of Deutsch's algorithm

Case 2 balanced mapping

$f(00) = f(11) = a$
 $f(01) = f(10) = b \rightarrow \overline{f(00)} = \overline{f(11)}$

⑧ $H_3 = [100\rangle (|f(00)\rangle - |f(00)\rangle) + |01\rangle (|f(00)\rangle - |f(00)\rangle) + |10\rangle (|f(00)\rangle - |f(00)\rangle) + |11\rangle (|f(00)\rangle - |f(00)\rangle)]$

$= (|00\rangle - |01\rangle - |10\rangle + |11\rangle) (|f(00)\rangle - |f(00)\rangle)$

$H_2 H_1 H_3 = |11\rangle (|f(00)\rangle - |f(00)\rangle)$

Summary

$|ab\rangle_{2,1} \rightarrow |ab\rangle_{1,2}$ (constant mapping)
 $|ab\rangle_{2,1} \rightarrow |ab\rangle_{2,1}$ (balanced mapping)





①

✓ - Deutsch algorithm
1-qubit register $\{0,1\} \rightarrow \{0,1\}$

✓ - Deutsch-Jozsa algorithm
n-qubit register $\{0,1\}^n \rightarrow \{0,1\}$

- optical implementation of Deutsch's algorithm

Deutsch algorithm (optical implementation)
classical illustration of Deutsch algorithm

① two-qubit state

✓ ② H-gate for qubit 1

③ H-gate for qubit 2

④ measurement for qubit 1

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

qubit 1: $\begin{cases} \rightarrow 11 \\ \uparrow 10 \end{cases}$

qubit 2: $\begin{cases} \rightarrow 11 \\ \uparrow 10 \end{cases}$

②

optical axis (for axis/containing in birefringent material)

phase

control

target

$E_{in} = |V\rangle \xrightarrow{\theta = 45^\circ} E_{out} = |H\rangle$

$E_{in} = |H\rangle \xrightarrow{\theta = 45^\circ} E_{out} = |V\rangle$

$E_{in} = |V\rangle \xrightarrow{\theta = 22.5^\circ} E_{out} = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$

$E_{in} = |H\rangle \xrightarrow{\theta = 22.5^\circ} E_{out} = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$

$E_{in} = |H\rangle \xrightarrow{\theta = 22.5^\circ + 22.5^\circ = 45^\circ} E_{out} = \frac{1}{\sqrt{2}}(|V\rangle - |H\rangle)$

X-gate ($\phi = 90^\circ$)

Z-gate ($\phi = 90^\circ$)

H-gate ($\phi = 67.5^\circ$)

H_2

open axis

closed axis



